Recovery of classical chaoticlike behavior in a conservative quantum three-body problem

M. J. Everitt*

Centre for Theoretical Physics, The British University in Egypt, El Sherouk City, Misr Ismalia Desert Road, Postal No. 11837, P.O. Box 43, Egypt

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Recovering trajectories of quantum systems whose classical counterparts display chaotic behavior has been a subject that has received a lot of interest over the last decade. However, most of these studies have focused on driven and dissipative systems. The relevance and impact of chaoticlike phenomena to quantum systems has been highlighted in recent studies which have shown that quantum chaos is significant in some aspects of quantum computation and information processing. In this paper we study a three-body system comprising of identical particles arranged so that the system's classical trajectories exhibit Hamiltonian chaos. Here we show that it is possible to recover very nearly classical-like, conservative, chaotic trajectories from such a system through an unravelling of the master equation. First, this is done through continuous measurement of the position of each system. Second, and perhaps somewhat surprisingly, we demonstrate that we still obtain a very good match between the classical and quantum dynamics by weakly measuring the position of only one of the oscillators.

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I. INTRODUCTION

Quantum mechanics is perhaps the most powerful and useful theory of physics to date. Indeed, with the possible emergence of many new quantum technologies in areas such as computation, communication, cryptography, and metrology this trend looks set to continue well into the future. With such strong interest in applications of quantum mechanics comes a concomitant interest in the measurement process and the interaction between "*classical*" and quantum systems. Indeed, as we wish to understand and apply quantum mechanics within the context of modern technology we will need to develop our understanding of what actually constitutes a classical device and how such objects interact with quantum systems. However, the recovery of classical mechanics is not always as simple as implied by a *naïveté* interpretation of the correspondence principle. This essential requirement of any physical theory can, for quantum mechanical systems, be stated as follows: "*If a quantum system has a classical analogue, expectation values of operators behave, in the limit* $\hbar \to 0$, *like the corresponding classical quantities*." [1]

We observe that interpretation of this statement can be problematic if, for example, we consider quantum systems that lack a specific dependence on Planks constant [2]. Further difficulties arise when attempting to recover the classical trajectories of classically nonlinear and chaotic systems as the Schrödinger equation is strictly linear.

We note that these concerns are no longer just of interest to those studying either the measurement problem or the correspondence principle and the emergence of the classical world. Indeed this area has a direct impact on quantum technologies. In order to fully leverage the power afforded by these emerging fields we must not only understand in depth the measurement process but also many-body quantum systems coupled to real environments. This is highlighted by the recent observation of chaos in the spectrum of Shor's algorithm [3] as well as in other studies involving quantum information processing and quantum chaos [4–6].

A solution to the correspondence problem for chaotic systems which has been employed with great success is found by utilizing quantum trajectories methods [7–17]. Here, introduction of environmental degrees of freedom and unravelling the master equation yield stochastic Schrödinger equations from which chaoticlike trajectories may be recovered. This process can be considered as comprising of several steps. First, we make the quantum system of an open one. This is archived by coupling the quantum object to an environment which may take the form of a measurement device. Once the environment has been introduced we model the evolution of the system's density operator (in the presence of the environment) using a linear master equation. However, master equations are similar to the Langevin equation insofar as they only predict a set of probable outcomes over an ensemble of systems or experiments. Therefore, in order to get some idea of the possible behavior of an individual experiment we next unravel the master equation. In essence, this process involves finding a stochastic differential equation for the system's state vector with the proviso that the dynamics given by the master equation are returned in the ensemble average over many solutions. There are infinitely many ways to do this each representing a different physical process. In this work we employ the quantum state diffusion (QSD) unravelling which corresponds to a unit-efficiency heterodyne measurement (or ambiquadrature homodyne detection) on the environmental degrees of freedom [16] (for a detailed introduction to this approach see [18]). Here the evolution of the state vector $|\psi\rangle$ is given by the Itô increment equation [13,14]

*Electronic address: m.j.everitt@physics.org

$$|d\psi\rangle = -\frac{i}{\hbar}H|\psi\rangle dt + \sum_j \left(\langle L_j^\dagger\rangle L_j - \frac{1}{2}L_j^\dagger L_j - \frac{1}{2}\langle L_j^\dagger\rangle\langle L_j\rangle\right)|\psi\rangle dt + \sum_j (L_j - \langle L_j\rangle)|\psi\rangle d\xi, \quad (1)$$

where the Lindblad operators L_i represent coupling to environmental degrees of freedom, dt is the time increment, and $d\xi$ are complex Weiner increments defined such that $\overline{d\xi^2} = \overline{d\xi} = 0$ and $\overline{d\xi d\xi^*} = dt$ [13,14]. Throughout this work a bar over a quantity denotes the average over stochastic processes while the notation $\langle\ \rangle$ is used for quantum mechanical expectation values. The first term on the right-hand side of this equation is the Schrödinger evolution of the system while the second (drift) and third (fluctuation) terms describe the decohering effects of the environment on the evolution of the systems state vector.

However, to date the body of work which uses quantum trajectories to recover classically chaoticlike trajectories has focused on those systems that are dissipative. There are several notable exceptions that demonstrate that continuous measurement of both driven and undriven conservative systems can recover classical-like behavior [19–22]. However, these works consider systems with only a single degree of freedom. Recently we became interested in whether it was possible, using a similar analysis, to recover chaotic trajectories of classical, multicomponent, systems undergoing Hamiltonian chaos. Initially we wished to consider the traditional three-body problem of classical mechanics for particles with similar masses [30]. This problem, although historically very significant, is nontrivial to solve. Consequently, in this work we consider a somewhat simplified system comprising of three coupled one-dimensional anharmonic oscillators.

II. BACKGROUND

The Hamiltonian for our chosen three-body system, comprising of one-dimensional anharmonic oscillators with a quartic potential and unit mass, is given by

$$H = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + \beta^2\left(\frac{q_1^2q_2^2 + q_2^2q_3^2 + q_1^2q_3^2}{2} + \frac{q_1^4 + q_2^4 + q_3^4}{32}\right). \quad (2)$$

The classical dynamics associated with this Hamiltonian can be chaotic and are known to have positive Lyapunov exponents [23]. When we consider classical mechanics q_i and p_i are taken to represent the classical values of position and momentum. However, when we consider the quantum mechanics they are replaced by their operator counterparts. As we shall always be clear as to which description we are considering at any one time this does not lead to any ambiguity.

We have already stated one expression of the correspondence principal in quantum mechanics. An alternative definition, which we find preferable, is to "*consider ħ fixed (it is) and scale the Hamiltonian so that the relative motion of the expectation values of the observable become large when compared with the minimum area* $(\hbar/2)$ *in the phase space*" [28]. In either case this is the role of the β term in the Hamiltonian, i.e., $\hbar \to \beta\hbar$ so that the smaller β the larger the dynamics when compared to a plank cell.

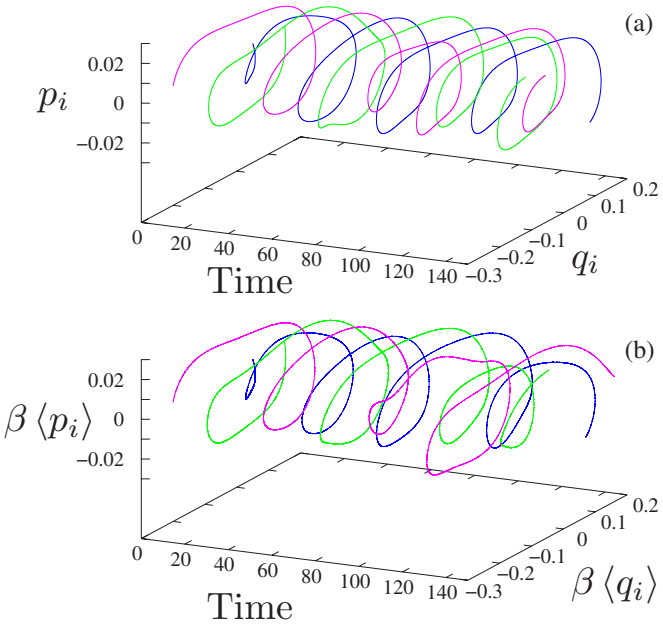

FIG. 1. (Color online) An example chaotic trajectory (a) solutions to the classical equations of motion (3) for the initial conditions $q_1 = -0.2$, $q_2 = 0.05$, $q_3 = 0.15$ for our chosen system and (b) scaled quantum expectation values $\beta\langle q_i\rangle$ and $\beta\langle p_i\rangle$ versus time for an unravelling of the master equation with initial state $D_1(-0.2/\beta)D_2(0.05/\beta)D_3(-0.15/\beta)|000\rangle$ and $\beta = 1/2000$. Trajectories for oscillator 1 are shown in (magenta) medium gray, for oscillator 2 in (blue) dark gray, and for oscillator 3 in (green) light gray. Note, all quantities are dimensionless.

III. RESULTS

From the Hamiltonian (2) we find the three classical equations of motion are

$$\ddot{q}_i + \beta^2\left(\frac{q_i^3}{8} + \sum_{i\neq j} q_i q_j^2\right) = 0, \quad \text{where } i,j = 1,2,3. \quad (3)$$

When we solve these coupled equations of motion with the initial conditions $q_1 = -0.2/\beta$, $q_2 = 0.05/\beta$, $q_3 = 0.15/\beta$, and $p_i = 0$ for all i we find that the dynamics are chaotic. We show the phase portrait for the solutions to these equations in Fig. 1(a) where, without loss of generality, we have set $\beta = 1$. We note as an interesting aside, one feature of this system is that if $q_i = p_i = 0$ for any i then $q_i = p_i = 0$ always.

We now proceed to discuss the quantum mechanical description of these coupled oscillators. Unlike the classical equations of motion the Schrödinger equation for this, or any other, system is strictly linear. We find that solutions to the Schrödinger equation for this system, even for moderate values of β, delocalize so rapidly that obtaining accurate solutions is not possible for us. This does, however, reinforce the lack of correspondence between classical dynamics and Schrödinger evolution for this system.

Following past work [7–17] on recovering classically chaoticlike orbits from a system's quantum counterpart we solve the unravelling of the master equation (1) with Hamiltonian (2). For this example there are three points of note with regard to possible choices of the environmental degrees of freedom. First, coupling to an environment helps localize the system's state vector and hence produce a well defined, classical-like, trajectory. Second, as the classical system is Hamiltonian and therefore conservative, we must choose the environment of each oscillator so that energy exchange is minimized between any part of the system and the environmental degrees of freedom. Third, we should specify a physically reasonable environment.

In this work we have chosen initially one of the most obvious candidates for the environment which satisfies all these conditions. Explicitly we have set each Lindblad $L_i = \kappa q_i$, $i=1,2,3$ corresponding to the continuous measurement of position. This unravelling also corresponds to that of the master equation for a weakly coupled, high temperature, thermal environment [19]. Here κ represents the magnitude of the coupling between each component of the system and its respective environment. In this work we use several values of κ. In Fig. 1(b) we use an intermediate coupling ($\kappa=0.1$) while in Figs. 2–5 we also present results for weak ($\kappa=0.01$) and strong ($\kappa=0.5$) couplings.

As our initial state, and for the best possible match with the classical initial conditions, we choose a tensor product of coherent states for which the quantum expectation values in position and momentum are centered in q-p phase plane at $q_1=-0.2/\beta$, $q_2=0.05/\beta$, $q_3=0.15/\beta$, and $p_i=0$ where $i=1,2,3$. Alternatively, we can express this initial condition explicitly as translated vacuum states by $D_1(-0.2/\beta)D_2(0.05/\beta)D_3(-0.15/\beta)|000\rangle$ where $D_i(\ )$ is the displacement operator in position for each component of the system. Here we have chosen $\beta=1/2000$ as this is the smallest value for which we can solve (1) both accurately and within a reasonable time frame. In order to help the reader quantify the time scale over which our results are presented we note that the logarithm time associated with our chosen value of beta is $\ln(1/\beta)\approx 3.3$ [24,25]. This is much shorter than the period over which we present the evolution of the system's trajectories.

In Fig. 1(b) we show the dynamics of the quantum expectation values of the position and momentum operators for each oscillator. These have been scaled by a factor of β so that they may be compared with Fig. 1(a). Here we see very good agreement initially, and similar characteristics throughout, the displayed dynamics. Indeed the trajectories are similar enough that it is impossible to determine from the graph alone which plot shows the classical and which the quantum evolution. We note that these curves begin to differ after a short period of time. However, this is not unexpected as the system we are analyzing is chaotic. In order to make the reasonable comparison of these results readily available we also include a graph of the evolution of q_1 and $\langle q_1\rangle$ as a function of time in Fig. 2 for three different couplings to the environment. It is apparent that there is a very good match between the quantum expectation values and the classical trajectory for $\kappa=0.01$.

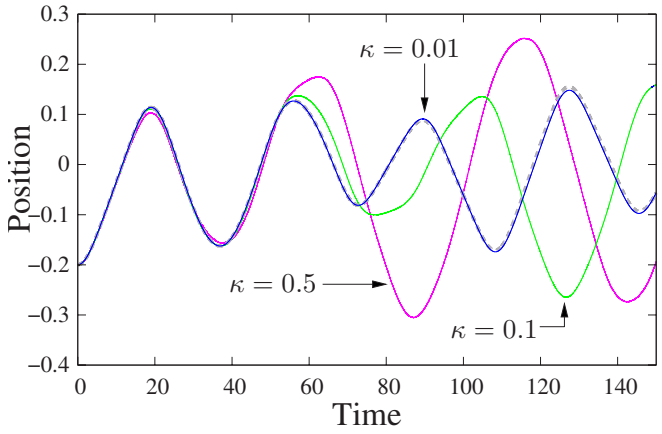

FIG. 2. (Color online) A comparison between the classical position q_1 (dashed gray) and $\beta\langle q_1\rangle$ for three different couplings (κ) to the environment. Note, all quantities are dimensionless.

We note that simply by including an environment our system no longer undergoes Hamiltonian evolution. In other words, in order to be able to recover classical-like trajectories of quantum systems whose classical counterparts exhibit Hamiltonian chaos we include environmental degrees of freedom that imply non-Hamiltonian evolution of the quantum system. However, using a sufficiently low coupling strength to the environmental results in a concomitant reduction both in energy exchange between the system and it's environment, and the localization of the state vector. We now verify that the solutions to Eq. (1) are, to good approximation, conservative. This does indeed appear to be the case for both intermediate and weak couplings ($\kappa=0.1$ and 0.01) but not for the stronger coupling ($\kappa=0.5$). This can be seen in Fig. 3 where we show the total energy found by substituting $\langle q_i\rangle$ and $\langle p_i\rangle$ for q_i and p_i into the Hamiltonian (2), i.e.,

$$\text{Energy} = \sum_i \frac{1}{2}\langle p_i\rangle^2 + \frac{\beta^2}{32}\langle q_i\rangle^4 + \frac{\beta^2}{2}\sum_{i\neq j}\langle q_i\rangle^2\langle q_j\rangle^2, \qquad (4)$$

where $i,j=1,2,3$. We note that we do not compute $\langle H\rangle$ as we wish to compare directly with the equivalent classical calculation.

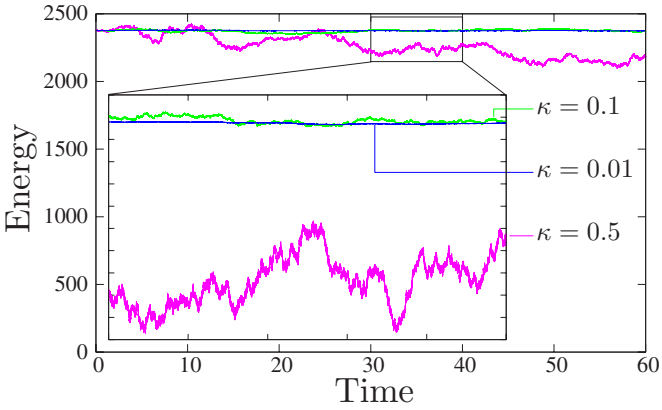

FIG. 3. (Color online) Total system energy, with magnified section inset, computed by substituting $\langle q_i\rangle$ and $\langle p_i\rangle$ into the Hamiltonian (2) for three different couplings (κ) to the environment. Note, all quantities are dimensionless.

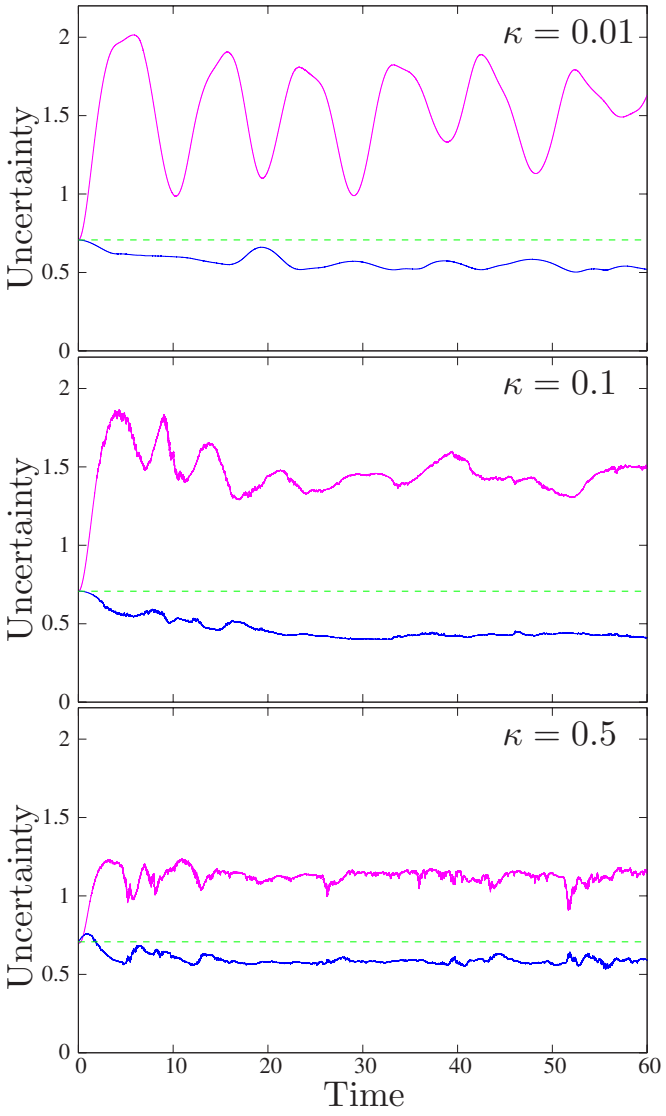

FIG. 4. (Color online) Uncertainty in position [light gray (magenta)] and momentum [dark gray (blue)] as a function of time for the first component for three different couplings (κ) to the environment. Values beneath the dashed (green) line indicate squeezing. Note, all quantities are dimensionless.

Next we verify localization of the state vector by computing the uncertainty in position and momentum for the first oscillator for three different couplings to the environment. Because both $\Delta q_i = \sqrt{\langle q_i^2\rangle - \langle q_i\rangle^2}$ and $\Delta p_i = \sqrt{\langle p_i^2\rangle - \langle p_i\rangle^2}$ between components behave in a similar fashion, we do not show results for the other two oscillators here. As is evident from Fig. 4 the interaction with the environment causes the system's state vector to localize within each of the component spaces. It is also apparent from this figure that the level of localization is dependent of the coupling between each of the system's components and their respective environments. We also note that as the system evolves it's states become squeezed in each of the momentum variables. Unlike the results presented in Fig. 1(b) we have not scaled these uncertainty values by $\beta = 1/2000$. Consequently, for direct comparison with the first two figures the results presented in Fig. 4 should be divided by 2000. Hence, the uncertainty in either position or momentum can be seen to be quite negligible when compared with the trajectory of quantum expectation values plotted in Fig. 1(b).

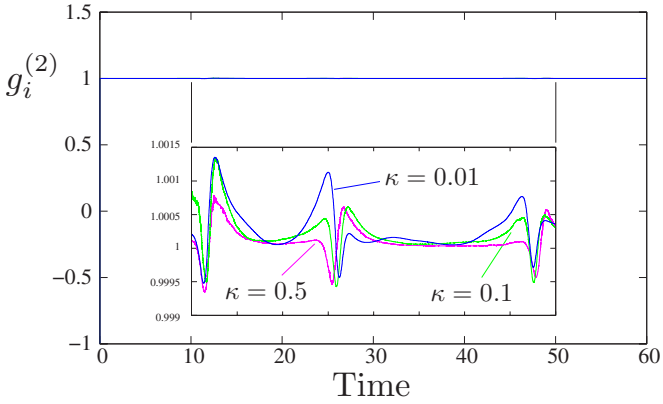

FIG. 5. (Color online) $g_i^{(2)}$ ($i=1$) coefficient, with magnified section inset, as a function of time for three different couplings (κ) to the environment. Classical-like motion yields $g_i^{(2)}=1$ as this implies a Poissonian statistics for the state of the system. Note, all quantities are dimensionless.

We can extract further information on the dynamics of this system simply by borrowing a technique from quantum optics. Namely through analyzing the photon statistics (bunching of photons) described by the second order correlations [26,27]

$$g_i^{(2)} = \frac{\langle n_i^2\rangle - \langle n_i\rangle}{\langle n_i\rangle^2}, \quad i = 1,2,3 \tag{5}$$

Where n is the number operator. Values of $g^{(2)}$ greater than 1 indicate photon bunching where photons arrive in groups while values of $g^{(2)}$ smaller than 1 indicate antibunching, a purely quantum mechanical phenomena representing precisely regular arrival of photons. However $g^{(2)}=1$ corresponds to Poissonian statistics and which is what we would expect from the state of our system should it be undergoing a classical like evolution. As we can see from Fig. 5 this is indeed the case (where, again, we have only shown data for the first component).

In the discussion above we have considered what happens when we perform simultaneous, continuous, measurements of the position of each of the components of the system. As the weak measurement limit is approached we find good agreement between classical and quantum dynamics and the system becomes, to good approximation, conservative. We now demonstrate that there exists a weaker condition under which we can produce the same outcome. That is, we show that weak measurement of only one of the position variables is sufficient to produce, to very good approximation, classical-like trajectories. As no one oscillator has a privileged status over the others we set, without loss of generality, $L_1 = \kappa q_1$, $L_2 = L_3 = 0$ with $\kappa = 0.01$ and $\beta = 1/2000$. Again we solve the unravelling of the master equation (1) with Hamiltonian (2) and the initial state $D_1(-0.2/\beta)D_2(0.05/\beta) \times D_3(-0.15/\beta)|000\rangle$. As we can see from Fig. 6 even under these conditions we recover quantum trajectories whose ex-

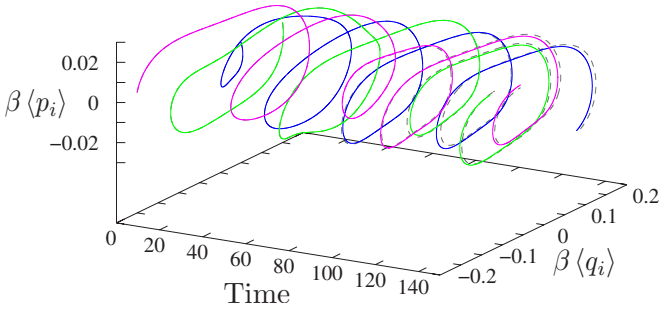

FIG. 6. (Color online) An example of chaoticlike trajectory for the scaled quantum expectation values $\beta\langle q_i\rangle$ and $\beta\langle p_i\rangle$, for weak measurement of the first oscillator only. Here $L_1=\kappa q_1, L_2=L_3=0$ for $\kappa=0.01$ and $\beta=1/2000$. Quantum trajectories for oscillator 1 are shown in (magenta) medium gray, for oscillator 2 in (blue) dark gray, and for oscillator 3 in (green) light gray. The corresponding classical dynamics are shown with a dashed gray line. Note, all quantities are dimensionless.

pectation values match very well indeed with those of the equivalent classical system.

IV. CONCLUSION

For any given β and an initially localized state there will be some agreement between the dynamics of the classical system and the evolution of the quantum expectation values of the corresponding quantum operators. However, after a short period of time the quantum state vector will begin to delocalize and differences between the predictions of each theory become apparent. We have demonstrated, by localizing the state vector through measurement of the position of one or all components of the system, that near classical-like dynamics can be recovered through unravelling the master equation. For this work we have chosen quantum state diffusion. However, it is likely that any other unravelling will produce similar results. Such detailed analysis is beyond the scope of this work and would belong in a more in depth study. We also note, as a subject for future study, that it would be interesting to determine the conditions under which measurement of a subset of the degrees of freedom of an N-body system results in the localization of the state vector.

Finally we would like to observe that following [28] it would be interesting to characterize the entanglement between the components of this system. As it may well be the case that for this example, as well as the one studied in [28], that the entanglement does not necessarily vanish in the classical limit. Unfortunately current restrictions on computational power prevent us from conducting this study at the present time. However, from the last result presented here we intuitively feel that there must persistently exist at least a small degree of entanglement between the first component and each of the other two. In order to justify this statement we propose the following argument. First, consider the extension to the tensor product space of the Lindblad operator, explicitly this is $\kappa q_1 \otimes 1_2 \otimes 1_3$. Now, by examining Eq. (1) we see that if the state of the system was separable, the last two terms (those responsible for the localization of $|\psi\rangle$) would not affect the components of the state vector for the second and third degrees of freedom. As introduction of this environmental degree of freedom localizes the state vector and result in the recovery of a classical-like trajectory, we therefore propose that the state vector must possess some nonzero entanglement.

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[30] We note that although the helium atom is also a standard example of a three-body system, due to the very large mass of the nucleus, this particular example simplifies to the restricted three-body problem.